

COHOMOLOGICAL DIMENSION, SELF-LINKING, AND SYSTOLIC GEOMETRY

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ABSTRACT. Given a closed manifold M , we prove the upper bound of

$$\frac{1}{2}(\dim M + \text{cd}(\pi_1 M))$$

for the number of systolic factors in a curvature-free lower bound for the total volume of M , in the spirit of M. Gromov's systolic inequalities. Here "cd" is the cohomological dimension. We apply this upper bound to show that, in the case of a 4-manifold, the Lusternik–Schnirelmann category is an upper bound for the systolic category. Furthermore we prove a systolic inequality on a manifold M with $b_1(M) = 2$ in the presence of a nontrivial self-linking class of a typical fiber of its Abel–Jacobi map to the 2-torus.

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Date: December 15, 2009.

2000 *Mathematics Subject Classification.* Primary 53C23; Secondary 55M30, 57N65.

Key words and phrases. Category weight, cohomological dimension, detecting element, Eilenberg–Ganea conjecture, essential manifolds, free fundamental group, Lusternik–Schnirelmann category, Massey product, self-linking class, systolic category.

¹Supported by the NSF, grant DMS-0604494.

²Supported by the Israel Science Foundation (grants 84/03 and 1294/06) and the BSF (grant 2006393).

³Supported by the NSF, grant 0406311.

1. SYSTOLIC INEQUALITIES AND LS CATEGORY

A quarter century ago, M. Gromov [Gr83] initiated the modern period in systolic geometry by proving a curvature-free 1-systolic lower bound for the total volume of an essential Riemannian manifold M of dimension d , i.e. a d -fold product of the systole is such a lower bound.

Here the term “curvature-free” is used in the literature to refer to a bound independent of curvature invariants, with a constant depending on the dimension (and possibly on the topology), but not on the geometry (i.e. the Riemannian metric). Note that such bounds cannot be called “metric-independent” as the systolic invariants themselves do depend on the metric.

Recently, M. Brunnbauer [Bru08c] proved that a $(k - 1)$ -connected manifold of dimension $n = kd$ satisfies a curvature-free stable k -systolic inequality $(\text{stsys}_k)^d \leq C \text{vol}_n$ if and only if a purely homotopic condition on the image of the fundamental class $[M]$ in a suitable classifying space is satisfied. Thus the total volume admits a lower bound in terms of a systolic product with d factors. J. Strom [St07] and the first named author [D08] were motivated by investigations of the Lusternik-Schnirelmann (LS) category [DKR08] in its relation to systolic geometry [KR06]. A. Costa and M. Farber [CF08] have pursued the direction initiated in [D08], as applied to motion planning-related complexity.

Our first result links systolic inequalities to the cohomological dimension (see [Bro94]) of the fundamental group, see Theorem 4.2 for a more precise statement.

1.1. Theorem. *Let M be a closed n -manifold, with $\text{cd}(\pi_1 M) = d \leq n$. Then there can be no more than*

$$\frac{n + d}{2}$$

systolic factors in a product which provides a curvature-free lower bound for the total volume of M .

It was shown in [KR06, KR08] that the maximal number of factors in such a product coincides with the LS category cat_{LS} in a number of cases, including all manifolds of dimension ≤ 3 . We apply Theorem 1.1 to show that, in dimension 4, the number of factors is bounded by the LS category, see Corollary 5.3 for a more precise statement.

1.2. Theorem. *For every closed orientable 4-manifold, the maximal number of factors in a product of systoles which provides a curvature-free lower bound for the total volume, is bounded above by $\text{cat}_{\text{LS}} M$.*

Combining Theorem 1.1 with a volume lower bound resulting from an inequality of Gromov's (see Sections 3 and 6), we obtain the following result (typical examples are $\mathbb{T}^2 \times S^3$ as well as the non-orientable S^3 -bundle over \mathbb{T}^2).

1.3. Theorem. *Let M be a closed 5-dimensional manifold. Assume that $b_1(M) = \text{cd}(\pi_1 M) = 2$ and furthermore that the typical fiber of the Abel–Jacobi map to \mathbb{T}^2 represents a nontrivial homology class. Then the maximal possible number of factors in a systolic lower bound for the total volume is 3. Note also that $3 \leq \text{cat}_{\text{LS}} M \leq 4$.*

The above result motivates the following question concerning upper bounds for the Lusternik–Schnirelmann category, cf. [D08].

1.4. Question. Under the hypotheses of Theorem 1.3, is the Lusternik–Schnirelmann category of M necessarily equal to 3?

It will be convenient to formulate all of the above results in terms of the systolic category. The idea of systolic category is to codify lower bounds for the total volume, in terms of lower-dimensional systolic invariants. We think of it as an elegant way of expressing systolic statements. Here we wish to incorporate all possible curvature-free systolic inequalities, stable or unstable. More specifically, we proceed as follows.

1.5. Definition. Given $k \in \mathbb{N}, k > 1$ we set

$$\text{sys}_k(M, \mathcal{G}) = \inf \left\{ \text{sysh}_k(M', \mathcal{G}; A), \text{stsys}_k(M, \mathcal{G}) \right\},$$

where sysh is the homology systole, stsys is the stable homology systole, and the infimum is over all regular covering spaces M' of M , and over all choices

$$A \in \{\mathbb{Z}, \mathbb{Z}_2\}. \quad (1.1)$$

Furthermore, we define

$$\text{sys}_1(M, \mathcal{G}) = \min \{ \text{sys}\pi_1(M, \mathcal{G}), \text{stsys}_1(M, \mathcal{G}) \}.$$

Note that the systolic invariants thus defined are nonzero [KR08].

1.6. Definition. Let M be a closed n -dimensional manifold. Let $d \geq 1$ be an integer. Consider a partition

$$n = k_1 + \dots + k_d, \quad k_1 \leq k_2 \leq \dots \leq k_d \quad (1.2)$$

where $k_i \geq 1$ for all $i = 1, \dots, d$. We say that the partition (or the d -tuple (k_1, \dots, k_d)) is *categorical* for M if the inequality

$$\text{sys}_{k_1}(\mathcal{G}) \text{sys}_{k_2}(\mathcal{G}) \dots \text{sys}_{k_d}(\mathcal{G}) \leq C(M) \text{vol}_n(\mathcal{G}) \quad (1.3)$$

is satisfied by all metrics \mathcal{G} on M , where the constant $C(M)$ is expected to depend only on the topological type of M , but not on the metric \mathcal{G} .

The *size* of a partition is defined to be the integer d .

1.7. Definition. The *systolic category* of M , denoted $\text{cat}_{\text{sys}}(M)$, is the largest size of a categorical partition for M .

In particular, we have $\text{cat}_{\text{sys}} M \leq \dim M$.

We know of no example of manifold whose systolic category exceeds its Lusternik-Schnirelmann category. The lower bound of $b_1(M) + 1$ for the systolic category of a manifold M with non-vanishing fiber class in the free abelian cover of M , discussed in Section 6, therefore inspires the following question.

1.8. Question. Is the non-vanishing of the fiber class in the free abelian cover of M , a sufficient condition to guarantee a lower bound of $b_1(M) + 1$ for the Lusternik-Schnirelmann category of M ?

The answer is affirmative if the fiber class can be represented as a Massey product, see Section 6.

The paper is organized as follows. In Sections 2 and 3, we review the notion of systolic category. In Section 4, we obtain an upper bound for the systolic category of a closed n -manifold in terms of its fundamental group and, in particular, prove that the systolic category of a 4-manifold with free fundamental group does not exceed 2 (Corollary 5.2).

In Section 6 we investigate the next possible value of the categories, namely 3. We recall a 1983 result of M. Gromov's related to Abel–Jacobi maps, and apply it to obtain a lower bound of 3 for the systolic category for a class of manifolds defined by a condition of non-trivial self-linking of a typical fiber of the Abel–Jacobi map. In fact, non-triviality of the self-linking class guarantees the homological non-triviality of the typical fiber lifted to the free abelian cover.

Marcel Berger's monograph [Be03, pp. 325–353] contains a detailed exposition of the state of systolic affairs up to '03. More recent developments are covered in [Ka07].

Recent publications in systolic geometry include the articles [AK10, Be08, Bru08a, Bru08b, Bru08c, BKS09, DKR08, 1, EL08, HKU08, KK08, KK09, KW09, Ka08, KS10, RS08, Sa08].

2. A SYSTOLIC INTRODUCTION

Let \mathbb{T}^2 be a 2-dimensional torus equipped with a Riemannian metric \mathcal{G} . Let A be the area of $(\mathbb{T}^2, \mathcal{G})$. Let ℓ be the least length of a

noncontractible loop in $(\mathbb{T}^2, \mathcal{G})$. What can one say about the scale-invariant ratio ℓ^2/A ? It is easy to see that the ratio can be made arbitrarily small for a suitable choice of metric \mathcal{G} . On the other hand, it turns out that the ratio is bounded from above. Indeed, C. Loewner proved that $\ell^2/A \leq 2/\sqrt{3}$, for all metrics \mathcal{G} on \mathbb{T}^2 , see [Pu52].

More generally, given a closed n -dimensional smooth manifold M with non-trivial fundamental group, M. Berger and M. Gromov asked whether there exists a constant $C > 0$ such that the inequality

$$\ell^n \leq C \operatorname{vol}(M) = C \operatorname{vol}_n(M, \mathcal{G}) \quad (2.1)$$

holds for all Riemannian metrics \mathcal{G} on M ; here ℓ is the least length of a noncontractible loop in M , while C is required to be independent of the metric \mathcal{G} . Indeed, Gromov [Gr83] proved that such a $C = C_n$ exists if M is an essential manifold, meaning that M represents a nonzero homology class in $H_n(\pi_1(M))$. I. Babenko [Ba93] proved a converse.

We generalize these invariants as follows. Let h_k denote the minimum of k -volumes of homologically nontrivial k -dimensional cycles (see Section 2 for details). Do there exist a partition $k_1 + \dots + k_d = n$ of n , and a constant C such that the inequality

$$\prod_{i=1}^d h_{k_i} \leq C \operatorname{vol}(M, \mathcal{G}) \quad (2.2)$$

holds for all Riemannian metrics \mathcal{G} on M ? The invariants h_k are the k -systoles, and the maximum value of d as in (2.2) is called the *systolic category* $\operatorname{cat}_{\operatorname{sys}} M$ [KR06, Ka07]. The goal of this work is to continue the investigation of the invariant $\operatorname{cat}_{\operatorname{sys}}$ started in [KR06, KR08].

It was originally pointed out by Gromov (see [Be93]) that this definition has a certain shortcoming. Namely, for $S^1 \times S^3$ one observes a “systolic freedom” phenomenon, in that the inequality

$$\operatorname{sysh}_1 \operatorname{sysh}_3 \leq C \operatorname{vol}(S^1 \times S^3) \quad (2.3)$$

is violated for any $C \in \mathbb{R}$, by a suitable metric \mathcal{G} on $S^1 \times S^3$ [Gr96, Gr99]. This phenomenon can be overcome by a process of stabilisation.

It turns out that the difference between the stable systoles, stsys_k , and the ordinary ones, sysh_k has significant ramifications in terms of the existence of geometric inequalities. Namely, in contrast with the violation of (2.3), the inequality

$$\operatorname{stsys}_1 \operatorname{stsys}_3 \leq \operatorname{vol}_4(S^1 \times S^3) \quad (2.4)$$

holds for all metrics on $S^1 \times S^3$ [Gr83] (see Theorem 3.2 for a generalisation).

3. GROMOV'S INEQUALITIES

Systolic category can be thought of as a way of codifying three distinct types of systolic inequalities, all due to Gromov. There are three main sources of systolic lower bounds for the total volume of a closed manifold M . All three originate in Gromov's 1983 Filling paper [Gr83], and can be summarized as follows.

- (1) Gromov's inequality for the homotopy 1-systole of an essential manifold M , see [We05, Gu09] and [Ka07, p. 97].
- (2) Gromov's stable systolic inequality (treated in more detail in [BK03, BK04]) corresponding to a cup product decomposition of the rational fundamental cohomology class of M , see Theorem 3.2.
- (3) A construction using the Abel–Jacobi map to the Jacobi torus of M (sometimes called the dual torus), also based on a theorem of Gromov (elaborated in [IK04, BCIK07]).

Let us describe the last construction in more detail. Let M be a connected n -manifold. Let $b = b_1(M)$. Let

$$\mathbb{T}^b := H_1(M; \mathbb{R})/H_1(M; \mathbb{Z})_{\mathbb{R}}$$

be its Jacobi torus. A natural metric on the Jacobi torus of a Riemannian manifold is defined by the stable norm, see [Ka07, p. 94].

The Abel–Jacobi map $\mathcal{A}_M : M \rightarrow \mathbb{T}^b$ is discussed in [Li69, BK04], cf. [Ka07, p. 139]. A typical fiber $F_M \subset M$ (i.e. inverse image of a generic point) of \mathcal{A}_M is a smooth imbedded $(n-b)$ -submanifold (varying in type as a function of the point of \mathbb{T}^b). Our starting point is the following observation of Gromov's [Gr83, Theorem 7.5.B], elaborated in [IK04].

3.1. Theorem (M. Gromov). *If the homology class $[\overline{F}_M] \in H_{n-b}(\overline{M})$ of the lift of F_M to the maximal free abelian cover \overline{M} of M is nonzero, then the total volume of M admits a lower bound in terms of the product of the volume of the Jacobi torus and the infimum of areas of cycles representing the class $[\overline{F}_M]$.*

We also reproduce the following result, due to Gromov [Gr83], see also [BK03, Ka07], stated in terms of systolic category.

3.2. Theorem (M. Gromov). *For a closed orientable manifold M , the systolic category of M is bounded from below by the rational cup-length of M .*

Note that Theorem 3.2 is not directly related to Theorem 3.1. For instance, Theorem 3.2 implies that the systolic category of $S^2 \times S^2$,

or \mathbb{CP}^2 , equals 2, while Theorem 3.1 gives no information about simply-connected manifolds.

4. FUNDAMENTAL GROUP AND SYSTOLIC CATEGORY

Throughout this section we will assume that π is a finitely presented group. Denote by $\text{cd}(\pi)$ the cohomological dimension of π , see [Bro94].

4.1. Lemma. *If $\pi_1(M)$ is of finite cohomological dimension then M admits a map $f : M \rightarrow B$ to a simplicial complex B of dimension at most $\text{cd}(\pi_1 M)$, inducing an isomorphism of fundamental groups.*

Proof. Let $\pi = \pi_1(M)$. When $d = \text{cd}(\pi) \neq 2$, by the Eilenberg–Ganea theorem [EG57], there exists a d -dimensional model $B\pi$ for the classifying space of π , which may be assumed to be a simplicial complex. This yields the desired map $f : M \rightarrow B\pi$ by universality of the classifying space.

In the case $d = 2$ it is still unknown whether one can assume that $\dim B\pi = 2$ for all π . That this is so is the content of the Eilenberg–Ganea conjecture. In this case we can assume only that $\dim B\pi \leq 3$.

We claim that there is a map $q : B\pi \rightarrow B\pi^{(2)}$ onto the 2-skeleton that induces an isomorphism of the fundamental groups. Indeed, the obstruction to retracting $B\pi$ onto $B\pi^{(2)}$ is an element of the cohomology group

$$H^3(B\pi; \pi_2(B\pi^{(2)}))$$

with coefficients in the π -module $\pi_2(B\pi^{(2)})$. This group is zero, since by hypothesis $\text{cd}(\pi) = 2$. By the classical obstruction theory the identity map of $B\pi^{(2)}$ can be changed on the 2-dimensional skeleton without changes on the 1-dimensional skeleton in such a way that a new map has an extension q to $B\pi$. Since $q : B\pi \rightarrow B\pi^{(2)}$ is the identity on the 1-skeleton, it induces an isomorphism of the fundamental groups.

Now in the case $d = 2$ we use the complex $B\pi^{(2)}$ instead of $B\pi$ together with the map $q \circ f : M \rightarrow B\pi^{(2)}$ instead of f . \square

4.2. Theorem. *Let M be a closed n -manifold, with $\text{cd}(\pi_1(M)) = d \leq n$. Then the systolic category of M is at most $(n + d)/2$.*

Proof. Let g_M be a fixed background metric on M . Choose a map

$$f : M \rightarrow B$$

as in Lemma 4.1, as well as a fixed PL metric g_B on B . Consider the metric $f^*(g_B)$ on M pulled back by f (see [Ba93, Ka07]). This metric is defined by a quadratic form of rank at most d at every point of M . Note that the quadratic form can be thought of as the *square* of the length element.

Next, we scale the pull-back metric by a large real parameter $t \in \mathbb{R}$. When the length of a vector is multiplied by t , the convention is to write the metric as $t^2 f^*(g_B)$. Since the rank of the metric is at most d , the volume of $g_M + t^2 f^*(g_B)$ grows at most as t^d , where g_M is any fixed background metric on M . We obtain a family of metrics

$$g_t = g_M + t^2 f^*(g_B) \quad (4.1)$$

on M with volume growing at most as t^d where $d = \dim B$, while the 1-systole grows as t . Thus the ratio

$$\frac{\text{sys}_1^{d+1}(M, g_t)}{\text{vol}(M, g_t)}$$

tends to infinity. The addition of a fixed background metric on M in (4.1) ensures a uniform lower bound for all its k -systoles for $k \geq 2$. It follows that a partition

$$n = \underbrace{1 + \cdots + 1}_{d+1} + k_{d+2} + \cdots + k_r$$

cannot be categorical. \square

5. SYSTOLIC CATEGORY, LS CATEGORY, AND $\text{cd}(\pi)$

5.1. Remark. The upper bound of Theorem 4.2 is sharp when π is a free group ($d = 1$) in the sense that π is the fundamental group of a closed $(2k + 1)$ -manifold

$$M = \left(\prod_{i=1}^{k-1} S^2 \right) \times (\#(S^1 \times S^2))$$

with $\text{cat}_{\text{sys}} M = k + 1 = (2k + 2)/2$ in view of Theorem 3.2. Similarly, for the $(2k + 2)$ -dimensional manifold

$$M = S^3 \times \prod_{i=1}^{k-2} S^2 \times (\#(S^1 \times S^2)),$$

where $k > 1$, we have $\text{cat}_{\text{sys}} M = k + 1$. The case of a 4-dimensional manifold follows from Corollary 5.2.

5.2. Corollary. *The systolic category of a closed orientable 4-manifold with free fundamental group is at most 2, and it is exactly 2 if M is not a homotopy sphere.*

Proof. The partitions $4 = 1 + 1 + 2$ and $4 = 1 + 1 + 1 + 1$ are ruled out by Theorem 4.2. The only remaining possibilities are the partitions $4 = 1 + 3$ (corresponding to category value 2) and $4 = 4$ (that would correspond to category value 1). If M is not simply-connected,

then by the hypothesis of the corollary, $b_1(M) \geq 1$ and therefore M satisfies a systolic inequality of type (2.4) (see Theorem 3.2), proving that $\text{cat}_{\text{sys}}(M) = 2$. If M is simply-connected but not a homotopy sphere, then $H_2(M) = \pi_2(M)$ is free abelian and $b_2(M) > 0$, so that M satisfies the systolic inequality

$$\text{stsys}_2(M)^2 \leq b_2(M) \text{vol}(M),$$

see [BK03] (a special case of Theorem 3.2), proving $\text{cat}_{\text{sys}}(M) = 2$. \square

5.3. Corollary. *For every closed orientable 4-manifold we have the inequality $\text{cat}_{\text{sys}} M \leq \text{cat}_{\text{LS}} M$, and the strict inequality is possible in the case $\text{cat}_{\text{sys}} M = 2 < 3 = \text{cat}_{\text{LS}} M$ only.*

We do not know, however, if the case of the strict inequality can be realized.

Proof. If the fundamental group of M is free then $\text{cat}_{\text{LS}} M \leq 2$ [MK95], and in this case the result follows from Corollary 5.2. If $\text{cat}_{\text{sys}} M = 4$ then $\text{cat}_{\text{LS}} M = 4$, [KR06, Ka07], and hence $\text{cat}_{\text{sys}} M \leq \text{cat}_{\text{LS}} M$ if $\text{cat}_{\text{LS}} M \geq 3$. Finally, if the fundamental group of M is not free then $\text{cat}_{\text{LS}} M \geq 3$ [DKR08]. \square

5.4. Remark. The equality $\text{cat}_{\text{sys}} M^n = \text{cat}_{\text{LS}} M^n$ for $n \leq 3$ was proved in [KR06, KR08].

6. SELF-LINKING OF FIBERS AND A LOWER BOUND FOR cat_{sys}

In this section we will continue with the notation of Theorem 3.1.

6.1. Proposition. *Let M be a closed connected manifold (orientable or non-orientable). If a typical fiber of the Abel–Jacobi map represents a nontrivial $(n - b)$ -dimensional homology class in \overline{M} , then systolic category satisfies $\text{cat}_{\text{sys}}(M) \geq b + 1$.*

Proof. If the fiber class is nonzero, then the Abel–Jacobi map is necessarily surjective in the set-theoretic sense. One then applies the technique of Gromov’s proof of Theorem 3.1, cf. [IK04], combined with a lower bound for the volume of the Jacobi torus in terms of the b -th power of the stable 1-systole, to obtain a systolic lower bound for the total volume corresponding to the partition

$$n = 1 + 1 + \cdots + 1 + (n - b),$$

where the summand “1” occurs b times. Note that Poincaré duality is not used in the proof. \square

The goal of the remainder of this section is to describe a sufficient condition for applying Gromov's theorem, so as to obtain such a lower bound in the case when the fiber class in M vanishes.

From now on we assume that M is orientable, has dimension n , and $b_1(M) = 2$. Let $\{\alpha, \beta\} \subset H^1(M)$ be an integral basis for $H^1(M)$. Let F_M be a typical fiber of the Abel–Jacobi map. It is easy to see that $[F_M]$ is Poincaré dual to the cup product $\alpha \smile \beta$. Thus, if $\alpha \smile \beta \neq 0$ then $\text{cat}_{\text{sys}} M \geq 3$ by Proposition 6.1. If $\alpha \smile \beta = 0$ then the Massey product $\langle \alpha, \alpha, \beta \rangle$ is defined and has zero indeterminacy.

6.2. Theorem. *Let M be a closed connected orientable manifold of dimension n with $b_1(M) = 2$. If $\langle \alpha, \alpha, \beta \rangle \smile \beta \neq 0$ then $\text{cat}_{\text{sys}} M \geq 3$.*

Note that, on the Lusternik–Schnirelmann side, we similarly have a lower bound $\text{cat}_{\text{LS}} M \geq 3$ if $\langle \alpha, \alpha, \beta \rangle \neq 0$, since the element $\langle \alpha, \alpha, \beta \rangle$ has category weight 2 [Ru97, Ru99].

To prove the theorem, we reformulate it in the dual homology language.

6.3. Definition. Let $F = F_M \subset M$ be an oriented typical fiber. Assume $[F] = 0 \in H_{n-2}(M)$. Choose an $(n-1)$ -chain X with $\partial X = F$. Consider another regular fiber $F' \subset M$. The oriented intersection $X \cap F'$ defines a class

$$\ell_M(F_M, F_M) \in H_{n-3}(M),$$

which will be referred to as the *self-linking class* of a typical fiber of \mathcal{A}_M .

The following lemma asserts, in particular, that the self-linking class is well-defined, at least up to sign.

6.4. Lemma. *The class $\ell_M(F_M, F_M)$ is dual, up to sign, to the cohomology class $\langle \alpha, \alpha, \beta \rangle \smile \beta \in H^3(M)$.*

Proof. The classes α, β are Poincaré dual to hypersurfaces $A, B \subset M$ obtained as the inverse images under \mathcal{A}_M of a pair $\{u, v\}$ of loops defining a generating set for $H_1(\mathbb{T}^2)$. The hypersurfaces can be constructed as inverse images of a regular point under a projection $M \rightarrow S^1$ to one of the summand circles. Clearly, the intersection $A \cap B \subset M$ is a typical fiber

$$F_M = A \cap B$$

of the Abel–Jacobi map (namely, inverse image of the point $u \cap v \in \mathbb{T}^2$). Then another regular fiber F' can be represented as $A' \cap B'$ where, say, the set A' is the inverse image of a loop u' “parallel” to u . Then $A' \cap X$ is a cycle, since

$$\partial(A' \cap X) = A' \cap A \cap B = \emptyset.$$

Moreover, it is easy to see that the homology class $[A' \cap X]$ is dual to the Massey product $\langle \alpha, \alpha, \beta \rangle$, by taking a representative a of α such that $a \smile a = 0$. Now, since $F' = A' \cap B'$, we conclude that $[F' \cap X]$ is dual, up to sign, to $\langle \alpha, \alpha, \beta \rangle \smile \beta$. \square

6.5. Remark. In the case of 3-manifolds with first Betti number 2, the non-vanishing of the self-linking number is equivalent to the non-vanishing of C. Lescop's generalization λ of the Casson-Walker invariant, cf. [Le96]. See T. Cochran and J. Masters [CM05] for generalizations.

Now Theorem 6.2 will follow from Theorem 6.6 below.

6.6. Theorem. *Assume $b_1(M^n) = 2$. The non-triviality of the self-linking class in $H_{n-3}(M)$ implies the bound $\text{cat}_{\text{sys}}(M) \geq 3$.*

The theorem is immediate from the proposition below. If the fiber class in M of the Abel–Jacobi map vanishes, one can define the self-linking class of a typical fiber, and proceed as follows.

6.7. Proposition. *The non-vanishing of the self-linking of a typical fiber $\mathcal{A}_M^{-1}(p)$ of $\mathcal{A}_M : M \rightarrow \mathbb{T}^2$ is a sufficient condition for the non-vanishing of the fiber class $[\overline{F}_M]$ in the maximal free abelian cover \overline{M} of M .*

Proof. The argument is modeled on the one found in [KL05] in the case of 3-manifolds, and due to A. Marin (see also [Ka07, p. 165–166]). Consider the pullback diagram

$$\begin{array}{ccc} \overline{M} & \xrightarrow{\overline{\mathcal{A}}_M} & \mathbb{R}^2 \\ p \downarrow & & \downarrow \\ M & \xrightarrow{\mathcal{A}_M} & \mathbb{T}^2 \end{array}$$

where \mathcal{A}_M is the Abel–Jacobi map and the right-hand map is the universal cover of the torus. Choose points $x, y \in \mathbb{R}^2$ with distinct images in \mathbb{T}^2 . Let $\overline{F}_x = \overline{\mathcal{A}}_M^{-1}(x)$ and $\overline{F}_y = \overline{\mathcal{A}}_M^{-1}(y)$ be lifts of the corresponding fibers $F_x, F_y \subset M$. Choose a properly imbedded ray $r_y \subset \mathbb{R}^2$ joining the point $y \in \mathbb{R}^2$ to infinity while avoiding x (as well as its \mathbb{Z}^2 -translates), and consider the complete hypersurface

$$S = \overline{\mathcal{A}}^{-1}(r_y) \subset \overline{M}$$

with $\partial S = \overline{F}_y$. We have $S \cap T_g \overline{F}_x = \emptyset$ for all $g \in G$, where $G \cong \mathbb{Z}^2$ denotes the group of deck transformations of the covering $p : \overline{M} \rightarrow M$ and T_g is the deck transformation given by g .

We will prove the contrapositive. Namely, the vanishing of the class of the lift of the fiber implies the vanishing of the self-linking class. If the surface \overline{F}_x is zero-homologous in \overline{M} , we can choose a compact hypersurface $W \subset \overline{M}$ with

$$\partial W = \overline{F}_x$$

(if there is no such hypersurface for \overline{F}_x , we work with a sufficiently high multiple NF_x , see [KL05] for details). The $(n-3)$ -dimensional homology class $\ell_M(F_x, F_y)$ in M can therefore be represented by the $(n-3)$ -dimensional cycle given by the oriented intersection $p(W) \cap F_y$. Now we have

$$p(W) \cap F_y = \sum_{g \in G} T_g W \cap \overline{F}_y = \sum_{g \in G} \partial(T_g W \cap S). \quad (6.1)$$

But the last sum is a finite sum of boundaries, and hence represents the zero homology class. The finiteness of the sum follows from the fact that the first sum contains only finitely many non-zero summands, due to the compactness of W . \square

To summarize, if the lift of a typical fiber to the maximal free abelian covering of M^n with $b_1(M) = 2$ defines a nonzero class, then one obtains the lower bound $\text{cat}_{\text{sys}}(M) \geq 3$, due to the existence of a suitable systolic inequality corresponding to the partition

$$n = 1 + 1 + (n - 2)$$

as in (1.3), by applying Gromov's Theorem 3.1.

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